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A TWO LINE TITLE
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CHAPTER 1

INTRODUCTION

1.1. Discussion of the Problems, Methods and Previous Results

In this dissertation we investigate a family of transcendental entire functions. We study the dynamics of these maps, the topology and the geometry of their Julia sets.

The dynamics in the class of transcendental entire functions, denoted usually by $\text{Ent}$ are different from the dynamics in the class of rational maps, for example, mainly because of the essential singularity at infinity and since the phase space is not compact. By the Picard theorem any neighborhood of infinity is mapped with infinite multiplicity over the entire plane missing at most one point. This particular situation makes the topology of the Julia set look very different from that of rational maps.

There is a class of entire maps whose dynamics is relatively well understood. This is the class $\mathcal{S}$ of entire functions that have finitely many asymptotic and critical values (maps of finite type) i.e. maps which have only finitely many singular values. More precisely, for every element $f \in \text{Ent}$, $\omega \in \hat{\mathbb{C}}$ is a singular value if $f$ is not a regular covering map over any neighborhood of $\omega$. The set of singular values is denoted by $\text{Sing}(f^{-1})$. Observe that, if $\omega$ is a non-singular value of $f$, then there exists a neighborhood $V$ of $\omega$ where every branch of $f^{-1}$ in $V$ is well defined and is a conformal map of $V$. This set $\text{Sing}(f^{-1})$ is very important in our investigation and the last observation is used extensively throughout this thesis.

That is, let $\mathcal{S}$ denotes the set $\{f \in \text{Ent} : \text{Sing}(f^{-1}) \text{is a finite set}\}$. $\mathcal{S}$ is usually called the class of finite singular type (transcendental) entire functions or the Speiser class. This class has been studied for many years and we refer the reader only to [13], [11] or [17]. Most of work in the field of iterates of transcendental entire functions has been centered around this class of maps and we also refer the reader for example to [13], [11], [21] [27], or [4]. This class $\mathcal{S}$ includes the 1-parameter exponential family $\{\lambda e^z\}_{\lambda \in \mathbb{C}^*}$ or the 1-parameter families
maps \{\lambda \sin z\}_{\lambda \in \mathbb{C}^*} \quad \text{and} \quad \{\lambda \cos z\}_{\lambda \in \mathbb{C}^*}. \quad \text{Maps from these last two families are conjugated to maps in the 2-parameter \textit{cosh} family \{ae^z + be^{-z}\}_{(a,b) \in \mathbb{C}^2}. In Chapters 1 and 2 of this thesis we shall recall also some of the most interesting properties of maps from these families.}

It is known that ([4, test], [17], [13]) the No Wandering Domains theorem holds for this type of maps and that the Julia set of these maps, satisfying additional conditions, contains Cantor Bouquets, as it was observed in [11]. Cantor bouquets are Cantor sets of curves called hairs. For more details on these topological objects the reader may consult for example [9], [11] or [1]. In section 1.4 we shall give the definition of a Cantor bouquet and we shall present some of the basic properties of this topological object. Another important observation (see for example [13]) is that maps in the class \mathcal{S} have the property that their Fatou set contains no Baker domain.

We shall present these and other fundamental properties of transcendental entire functions in Chapters 1 and 2. In Chapter 1 we will also discuss the previous results obtained in [27] and [28] by M.Urbański and A.Zdunik who studied expanding mappings \(f_\lambda(z) = \lambda e^z\) that have an attracting periodic orbit. They developed the thermodynamic formalism for some special \textit{potentials} associated with these functions. The present work is an answer to the question about the possibility of developing a similar theory for the \textit{cosh family}. The fundamental difference between the exponential family and the cosine family is that a map from the second one has no asymptotic values (only critical values) and a map from the other has only one singular value, the asymptotic value at 0.

In Chapter 1, sections 1.3 and 1.4, we recall some of the basic properties we need on the Dynamics of transcendental entire functions, we define the Speiser class and then we consider Cantor bouquets, following the work of Devaney, Krych, Tangerman, Aarts and Oversteegen, via the fundamental concept of \textit{straight brush} introduced by Aarts and Oversteegen in [1] in 1991.

In Chapter 2 we define the family \(\mathcal{H} \subset \mathcal{S}\) and we will establish its basic dynamical properties. We discuss the topology of the Julia set of these maps and we shall observe that the Julia set of these maps is a Cantor bouquet. In section 2.2 we prove the uniformly
expanding property and then in section 2.4 we study conjugacies in the family $\mathcal{H}$. Next, in Chapter 3 we build a version of thermodynamic formalism for maps in $\mathcal{H}$ and we show, among others, the existence and uniqueness of a $(t, \alpha)$—conformal measure for maps in the family, so then in section 3.3 it can be proven a Bowen’s type formula. In Chapter 4 it is shown that the Hausdorff dimension of the set of points in the Julia set of $f_a$ having non-escaping orbits (denoted by $J_{f_a}^r$) depends analytically on the parameter $a \in \mathbb{C}^{n+1}$.

In what follows we discuss the problems we shall study in this dissertation. There are two basic problems in iteration theory. The first, classical one is to study the iterative behavior of a single function. The second one is to study families of functions, especially how the dynamical behavior of a member in the family changes if the function is perturbed. The simplest (but already sufficiently complicated) case being a family of functions depending on one parameter (see for example the cases of exponential or sine families). A good understanding of the dynamics of an individual function is of course necessary for the study of problems involving perturbation of functions.

Mathematical models for phenomena in the natural sciences often lead to iteration of functions. But in what follows we study iteration theory for its own. Iteration theory of functions in one complex variable (or Holomorphic Dynamics) essentially originated with the work of Fatou [15] and Julia [18] at the beginning of the last century. At the same time, the iteration of rational functions was also investigated by Ritt [25]. In 1926 Fatou [15] extended some of the results to the case of transcendental entire functions. Julia did not consider the iteration of transcendental functions. In the last 30 years there was a renewed interest in the iteration theory of Holomorphic functions. Nowadays there exist many introductory books in the field of Complex Dynamics. We mention only [5], [22] and [23]. There are comparatively few expositions of the Dynamics of transcendental entire functions. We refer to [10] for the iteration of the exponential function and we refer again to [23] which has a chapter on the Dynamics of transcendental entire functions. But many papers were written on the Dynamics of transcendental entire maps and we will mention some of them whenever the research conducted there will interfere with our present work. In this thesis we answer
mainly to the following six questions. First question is about the existence and uniqueness of a \((t, \alpha)\)-measure for maps in the family \(\mathcal{H}\). Second we ask the question if it is possible to prove a Bowen’s type formula for these maps. The third interesting question is about the behavior of maps in the family under conjugating maps. The fourth, most important problem, is about the possibility of developing a thermodynamic formalism for maps in the family. The fifth important problem is on perturbation theory for maps in \(\mathcal{H}\). This question will direct us to this fundamental sixth question: if we perturb a little bit the parameters on which a map in the family depend on, how the Hausdorff dimension of the Julia set is changed?

**Previous Results and Discussion of the Methods**

The research conducted in [27], [28], [26] between 1999 and 2003 by Mariusz Urbański and Anna Zdunik motivated essentially our present work. In these papers, Urbański and Zdunik investigated the fractal geometry, the dynamics and the thermodynamic formalism of maps in the exponential family \(f_\lambda(z) = \lambda e^z\). They considered both hyperbolic and non-hyperbolic situations, and they proved first the existence and uniqueness of a probability conformal measure (with an exponent greater than 1) for some maps associated with maps in the exponential family (simply, these maps are projections of exponential maps on the periodicity strips of height \(2\pi i\)).

Then they proved various dynamical related properties, including a Bowen’s type formula and the fact that the Hausdorff dimension of the complement (in the Julia set \(J_{f_\lambda}\)) of the set of points escaping to infinity under forward iterates of \(f_\lambda\), is less than 2. This set was denoted by \(J'_{f_\lambda}\).

Next, for the hyperbolic situation, considering the parameters \(\lambda\) such that \(f_\lambda\) has an attracting periodic orbit (this family was denoted by \(Hyp\)) it is known that the Julia set of a map in \(Hyp\) is a Cantor Bouquet (see Chapter 2). In [28] Urbański and Zdunik studied perturbations in the exponential family and then, with the methods of thermodynamic formalism, they showed that the function \(\lambda \to HD(J'_{f_\lambda})\) is real-analytic. They proved also
that maps from $Hyp$ are uniformly expanding on their Julia set and they defined appropriately the topological pressure for some potentials associated to these maps, Perron-Frobenius operators with some more general potentials, and generalized (Gibbs) measures. It is important to observe that, for these maps (in contrast to the case of subshifts of finite type or distance expanding maps), among other difficulties, the phase space is not compact, the potentials are unbounded, and Perron-Frobenius operators are expressed as infinite series of other appropriate operators.

The special methods used by Urbański and Zdunik for the analysis of the dynamics entire transcendental functions are, as we already mentioned, the methods of Thermodynamic formalism. A question frequently asked is why the name “thermodynamic formalism”? Although the analogies are formal rather than physical, many of the ideas, such as the existence of Gibbs measures, were originally developed in statistical mechanics, and translated to dynamical systems many years later (refer to [14] for more comments).

The pair mathematics-physics is historically inseparable, with mathematics serving to “model physical reality with the intent to rationally understand and clearly expose its laws” (refer to [29]). Thermodynamics is a science created by (among others) Boltzmann, Carnot, Kelvin, Maxwell, and Gibbs. This physical science gave birth to the (mathematical) theory of dynamical systems. Important to mention that the thermodynamic formalism generalized to the case of rational maps whose Julia set contains no critical points is due to Denker and Urbański (see [8]). Thermodynamic formalism applied to transcendental entire functions was also initiated, as we already observed, by Mariusz Urbański and Anna Zdunik, with the sequence of papers [27], [28], [26] published by the two authors between 2001 and 2003.

1.2. The Speiser class and Cantor Bouquets

In this section we follow the expositions from [6] and [23] on the Dynamics of transcendental entire functions. We are going to collect the very basic but fundamental facts which we need later in our exposition. We recall basic definitions and theorems emphasizing the class of transcendental entire functions of finite singular type (or the Speiser class). First we consider topological properties of the Fatou sets and the Julia sets of these maps. The
point at infinity is essential singularity of a transcendental entire function. The Picard theorem shows that, in a neighborhood of the point at infinity, the action of such a function is strongly “explosive”. Thus, in general, iteration of transcendental entire functions is much more complicated than that of polynomials. For an arbitrary polynomial, the point at infinity is always a superattracting fixed point. Hence the Julia set of a polynomial is compact in \( \mathbb{C} \). On the other hand we have the following.

**Proposition 1.2.1 (Julia set is unbounded).** The Julia set of a transcendental entire function is unbounded in \( \mathbb{C} \).

**Proof.** Let \( f \) be a transcendental entire function. Choose a point \( \omega \) in Julia set \( J_f \) which is not exceptional value in the sense of Picard. Let \( U \) be any neighborhood of the point at infinity. The Picard theorem shows that there is a point \( \theta \) in \( U \) such that \( f(\theta) = \omega \). Since the Julia set is backward invariant, \( \theta \) is in \( J_f \). Thus \( J_f \) is unbounded. \( \square \)

Again recall that, in the case of a polynomial, the immediate basin of attraction of the point at infinity is completely invariant. Hence its boundary is the Julia set of the polynomial. It follows that all Fatou components except for this basin are simply connected.

**Proposition 1.2.2 (No Herman rings).** Let \( f \in \text{Ent} \cup \text{Poly} \), where by \( \text{Ent} \) we denoted the set of transcendental entire functions and by \( \text{Poly} \) we denote the set of polynomial functions. Then its Fatou set contains no cycles of Herman rings.

**Proof.** Suppose that there were a Herman ring \( H \). Since \( \{f^n\}_{n=1}^\infty \) is uniformly bounded on \( H \), the maximum principle shows that \( \{f^n\}_{n=1}^\infty \) is uniformly bounded also in the bounded component \( U \) of \( \mathbb{C} - H \). This is a contradiction because \( U \) contains points of \( J_f \). \( \square \)

**Remark 1.2.3.** The above Proposition does not imply non-existence of multiply connected components of the Fatou set for transcendental entire functions. Baker was the first to give an example of a multiply connected Fatou component (refer to [23]-Th 3.4.1). Also recall that (see for example Theorem 3.15 in [23]) every unbounded Fatou component of a map in
Ent is simply connected and, as a corollary, observe that the Julia set of a map in Ent is never totally disconnected.

**Definition 1.2.4 (Singular values).** For every \( f \in \text{Ent} \) we call \( \alpha \in \hat{\mathbb{C}} \) a singular value if \( f \) is not a smooth covering map over any neighborhood of \( \alpha \). We denote the set of all singular values by \( \text{Sing}(f^{-1}) \).

If \( \alpha \) is a non-singular value of \( f \), then there exists a neighborhood \( V \) of \( \alpha \) where every branch of \( f^{-1} \) in \( V \) is well defined and is conformal map of \( V \).

**Definition 1.2.5 (Eremenko-Lyubich).** We call a transcendental entire function to be of finite singular type or to belong to the Speiser class if it belongs to \( S \) where

\[
S = \{ f \in \text{Ent} : \text{Sing}(f^{-1}) \text{ is a finite set} \}.
\]

Observe that

\[
S = \{ f \in \text{Ent} : \text{the set of critical and asymptotic values is finite} \}.
\]

Recall that the set of critical points of a function \( f \) is defined by:

\[
\text{Crit}(f) = \{ z : f'(z) = 0 \}
\]

and the set of critical values is \( f(\text{Crit}(f)) \). Also we make the following observation.

**Remark 1.2.6.** For \( f \) a polynomial or rational function the dynamics is determined in large measure by the behavior of orbits of the critical values. For \( f \in \text{Ent} \), the set of singular orbits must be extended as we can see for \( f_a(z) = ae^z \) with \( 0 < a < \frac{1}{e} \), \( f_a \) has no critical points but the essential role is played by \( 0 \) which is an omitted value of \( f_a \). In fact \( 0 \) is an asymptotic value.

For a map \( f \in \text{Ent} \) a point \( w \in \overline{\mathbb{C}} \) is an asymptotic value for \( f \) if there is a continuous curve \( \gamma(t) \) (called a path of determination) satisfying

\[
\lim_{t \to \infty} \gamma(t) = \infty
\]
and
\[ \lim_{t \to \infty} f(\gamma(t)) = w. \]

Any curve which tends to \( \infty \) such that \( \text{Re } z \to -\infty \) is such a curve \( \gamma \) for \( f_a \) (take for example \( \gamma(t) = -t^2 + it \)) so 0 is an asymptotic value for \( f_a \).

A Picard exceptional value (omitted value) is an asymptotic value for an entire function (see 0 and \( \infty \) for \( e^z \) which are both omitted values).

We observe also that there is a dichotomy in the Speisser class as there exists for quadratic polynomials, where there are basically two types of Julia sets, Cantor sets and Julia sets that are connected; for maps in the Speiser class there is a similar dichotomy, either Julia set is \( \mathbb{C} \) or Julia set is a Cantor bouquet.

**Theorem 1.2.7** (Fundamental Theorem). If \( f \in S \) then the Fatou set \( F_f \) contains no wandering domains, no Baker domains and every component of \( F_f \) is simply connected.

For the proof we refer the reader to [23] or [6].

We also recall that the set of repelling periodic points is dense in the Julia set. Hence the set of the points whose orbits are bounded is dense in the Julia set. Next we discuss the relation between the Julia set \( J_f \) and the set of *escaping* points (see [6] p.26).

\[ I_f = \{ z \in \mathbb{C} : \lim_{n \to \infty} f^n(z) = \infty \}. \]

If \( f \in Poly \) then \( I_f \) is the immediate attractive basin of the superattracting fixed point \( \infty \). In this case it was proved that \( J_f = \partial I_f \). Eremenko showed first that if \( f \in Ent \) then \( I_f \) is not an empty set and next he showed that if \( f \in S \) then \( J_f = \overline{I_f} \). Eremenko asked then two fundamental questions: Is every component of \( I_f \) unbounded? Can every point in \( I_f \) be joined with \( \infty \) by a curve in \( I_f \)? Clearly a positive answer to the second question will imply that the answer to the first question is also positive. Eremenko proved that \( \overline{I_f} \) does not have bounded components and he remarked that a positive answer to the second question for a restricted class of functions follows from the results of Devaney and Tangerman (see [11]).

We are going to present the results of Devaney and Tangerman and some others about Cantor bouquets in the next section.

In a paper from 1986, [11], Devaney and Tangerman showed that maps in the Speiser class satisfying some growth conditions admit “Cantor bouquets” in their Julia set. All of the curves (hairs) in the bouquet tend to $\infty$ in the same direction, and the map behaves like the shift automorphism on the Cantor set. Hence the dynamics near $\infty$ for these maps may be analyzed completely. Among the maps in $\text{Ent}$, for which the Devaney and Tangerman methods apply, are $\exp(z), \sin(z), \cos(z), \cosh(z), \sinh(z)$ and we will see that maps in the more general family $\mathcal{H}$ (that we deal with in the research presented in this dissertation) also satisfy Devaney-Tangerman conditions and the Julia set of these maps is itself a Cantor bouquet.

Cantor bouquets arise very often in the dynamics of maps in $\mathcal{S}$. Examples include maps for the exponential family (see [10] or [9]) $f_\lambda = \lambda e^z$ for parameters $\lambda$ satisfying $0 < \lambda < \frac{1}{e}$ or maps in the sine family $f_\lambda = \lambda \sin z$ with $\lambda$ real satisfying $0 < \lambda < 1$. Also (refer to [1]) it was shown that Julia sets of maps in the one parameter families $\{ \lambda \cosh z \}$ with $0 < \lambda < 0.67$ and $\{ \lambda \sinh z \}$ for $0 < \lambda < 0.85$ are also Cantor bouquets. We do not go into details in this thesis, but in my paper [7], which is now in preparation, I shall treat these problems with an accent on the family $\mathcal{H} \subset \mathcal{S}$ which will be defined in the next section. Here we just want to give a rigorous definition of a Cantor bouquet following [1] and [9]. To describe the topological structure of a Cantor Bouquet, we need to introduce the notion of a straight brush.

To each irrational number $\zeta$, we assign (there are many ways to do this (see for example [9])) an infinite string of integers $n_0 n_1 n_2 \ldots$ as follows. We will break up the real line into open intervals $I_{n_0 n_1 \ldots n_k}$ which have the following properties

(i) $I_{n_0 \ldots n_k+1} \subset I_{n_0 \ldots n_k}$.

(ii) The endpoints of $I_{n_0 \ldots n_k}$ are rational.

(iii) $\zeta = \cap_{k=1}^{\infty} I_{n_0 \ldots n_k}$. 

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**Definition 1.2.8** (Straight brush-Aarts/Oversteegen). A straight brush $B$ is a subset of $[0, \infty) \times \mathcal{N}$, where $\mathcal{N}$ is a dense subset of irrationals, having the following three properties

(i) $B$ is “hairy” in the following sense. If $(y, \alpha) \in B$, then there exists a $y_\alpha \leq y$ such that $(t, \alpha) \in B$ iff $t \geq y_\alpha$. That is the “hair” $(t, \alpha)$ is contained in $B$ where $t \geq y_\alpha$. And $y_\alpha$ is called the endpoint of the hair corresponding to $\alpha$.

(ii) Given an endpoint $(y_\alpha, \alpha) \in B$ there are sequences $\beta_n \uparrow \alpha$ and $\gamma_n \downarrow \alpha$ in $\mathcal{N}$ such that $(y_{\beta_n}, \beta_n) \rightarrow (y_\alpha, \alpha)$ and $(y_{\gamma_n}, \gamma_n) \rightarrow (y_\alpha, \alpha)$. That is, any endpoint of a hair in $B$ is the limit of endpoints of other hairs from both above and below.

(iii) $B$ is closed subset of $\mathbb{R}^2$.

We observe that a straight brush is a remarkable topological object and we view it as a subset of the Riemann sphere. Aarts and Oversteegen have shown that any two straight brushes are ambiently homeomorphic, i.e. there is a homeomorphism of $\mathbb{R}^2$ taking one brush onto another. This important observation led to the formal definition of the Cantor bouquet.

**Definition 1.2.9.** A Cantor bouquet is a subset of $\overline{\mathbb{C}}$ that is homeomorphic to a straight brush with $\infty$ mapped to $\infty$.

We shall make the observation that the Julia set of a map in the family $\mathcal{H}$, which we are going to define in the next Chapter, is a Cantor bouquet. We shall see that its existence follows from [11] and then we shall conclude that with Aarts and Oversteegen’s methods, it can be shown that the Julia set of a map in $\mathcal{H}$ is indeed homeomorphic to a straight brush. We mention that a more extensive approach is done in my paper [7] which is in preparation and it will be ready to be submitted for publication in the near future.
CHAPTER 2
THE FAMILY $\mathcal{H}$

2.1. Definition of $\mathcal{H}$ and the Uniformly Expanding Property

In this section we define the family $\mathcal{H}$ and we establish basic dynamical properties of a map $f_a \in \mathcal{H}$. Then we prove the important Lemma 2.1.1.

Definition of $\mathcal{H}$

We define the family $\mathcal{H}$ as a family of maps in the Speiser class of transcendental entire functions of finite singular type.

Let $a = (a_0, a_1, \ldots, a_n) \in \mathbb{C}^{n+1}$ be a vector such that $a_0 \neq 0$, $a_n \neq 0$,

$$P_a(z) = a_n z^n + \cdots + a_1 z + a_0 \in \mathbb{C}[z]$$

and

$$g_a(z) = \frac{P_a(z)}{z^k}$$

where $k$ is a positive integer strictly less than $n = \deg(P_a) \geq 2$. Define

$$f_a(z) = g_a \circ \exp(z) = \frac{a_n e^{nz} + a_{n-1} e^{(n-1)z} + \cdots + a_1 e^z + a_0}{e^k} = \sum_{j=0}^{n} a_j e^{(j-k)z}$$

Observe that maps of this form do not have any finite asymptotic values. This is the reason why we restricted ourselves to integers $k$ satisfying condition $0 < k < n$. As it was mentioned in Chapter 1, the most well known examples of this type of maps are maps from the cosine family.

We denote by $\text{Crit}(f_a)$ the set $\{z : f_a'(z) = 0\}$. Observe that

$$f_a'(z) = \sum_{j=0}^{n} a_j (j-k) e^{(j-k)z}$$
and that \( g'_a(z) = 0 \) if and only if \( z P'_a(z) - k P_a(z) = 0 \), which is equivalent to
\[
\sum_{j=0}^{n} a_j(j-k)z^j = 0.
\]

Therefore, there exist \( n \) non-zero complex numbers (counting multiplicities) \( s_1, s_2, \ldots, s_n \) such that \( z \in \text{Crit}(f_a) \) if and only if \( e^z = s_k \) for some \( k = 1, 2, \ldots, n \) i.e.
\[
\{ z_k = \log s_k + 2\pi i m : m \in \mathbb{Z}, k = 1, \ldots, n \}
\]
is the set of critical points and observe that the set of critical values of a map \( f_a \) is finite.

Denote by \( \mathcal{H} \) the family of functions
\[
\mathcal{H} = \left\{ f_a(z) = \frac{P_a(e^z)}{e^{kz}} : \deg P_a > k > 0 \text{ and } \delta_a > 0 \right\},
\]
where by \( \mathcal{P}_{f_a} \) we denote the post-critical set of \( f_a \) i.e. the set
\[
\mathcal{P}_{f_a} = \bigcup_{n \geq 0} f^n_a(\text{Crit}(f_a))
\]
and
\[
\delta_a = \frac{1}{2} \min \left\{ \frac{1}{2}, \text{dist}(J_{f_a}, \mathcal{P}_{f_a}) \right\},
\]
where
\[
\text{dist}(J_{f_a}, \mathcal{P}_{f_a}) = \inf \{|z_1 - z_2| : z_1 \in J_{f_a}, z_2 \in \mathcal{P}_{f_a}\}
\]
is the Euclidean distance between the Julia set of \( f_a \), \( J_{f_a} \), and the post-critical set of \( f_a \), \( \mathcal{P}_{f_a} \).

The reason we define \( \delta_a \) in such a way will be more visible later on, starting with Chapter 3, and is due to the application (we shall need) of the Koebe Distortion Theorem since one can observe that, for every \( y \in J_{f_a} \) and for every \( n \geq 1 \), there exists a unique holomorphic inverse branch
\[
(f^n_a)^{-1}_y : B(f^n_a(y), 2\delta_a) \to \mathbb{C}
\]
such that \( (f^n_a)^{-1}_y \circ (f^n_a)(y) = y \).

Then there exists a numerical constant \( K \) such that, for \( z_1, z_2 \in J_{f_a} \) with \( |z_1 - z_2| < \delta_a \) and for \( y \in f^{-n}_a(z_1) \),
\[
\frac{1}{K} \leq \frac{|((f^n_a)^{-1}_y)'(z_1)|}{|((f^n_a)^{-1}_y)'(z_2)|} \leq K.
\]
Observe that $Crit(f_a) \subset F_{f_a}$, where $F_{f_a}$ is the Fatou set of $f_a$. Consequently, maps in the family $\mathcal{H}$ do not have neither parabolic domains nor Herman rings nor Siegel disks. Moreover, as was written in Chapter 1 they do not have neither wandering nor Baker domains. Also for every point $z$ in the Fatou set there exists (super)attracting cycle such that the trajectory of $z$ converges to this cycle.

**The Cylinder and the Definition of $J_{F_a}^r$**

Since the map $f_a \in \mathcal{H}$ is periodic with period $2\pi i$, we consider it on the quotient space $P = \mathbb{C}/\sim$ (the cylinder) where

$$z_1 \sim z_2 \text{ iff } z_1 - z_2 = 2k\pi i \text{ for some } k \in \mathbb{Z}.$$ 

If $\pi: \mathbb{C} \to P$ is the natural projection, then, since the map $\pi \circ f_a: \mathbb{C} \to P$ is constant on equivalence classes of relation $\sim$, it induces a holomorphic map

$$F_a: P \to P.$$

The cylinder $P$ is endowed with Euclidean metric which will be denoted in what follows by the same symbol $|w - z|$ for all $z, w \in P$. The Julia set of $F_a$ is defined to be

$$J_{F_a} = \pi(J_{f_a})$$

and observe that

$$F_a(J_{F_a}) = J_{F_a} = F_a^{-1}(J_{F_a}).$$

We shall study the set $J_{f_a}^r$ consisting of those points of $J_{f_a}$ that do not escape to infinity under positive iterates of $f_a$. In other words, if

$$I_\infty(f_a) = \{z \in \mathbb{C} : \lim_{n \to \infty} f_a^n(z) = \infty\},$$

then

$$J_{f_a}^r = J_{f_a} \setminus I_\infty(f_a)$$

and, if

$$I_\infty(F_a) = \{z \in P : \lim_{n \to \infty} F^n(z) = \infty\},$$
then

\[ J_{F_a}^r = J_{F_a} \setminus I_\infty(F_a). \]

In what follows we fix \( a \in \mathbb{C}^{n+1} \) and we denote for simplicity \( f_a \in \mathcal{H} \) by \( f \). The following Lemma reveals some background information for a better understanding of the dynamical behavior of maps in our family \( \mathcal{H} \). This lemma will be used several times and it will be a key technical ingredient for many proofs.

Observe first that, if we consider \( a = (a_0, \ldots, a_n) \in \mathbb{C}^{n+1} \), since

\[ f_a(z) = \sum_{j=0}^{n} a_j e^{(j-k)z} \]

we have

\[ f_a'(z) = \sum_{j=0}^{n} a_j (j-k) e^{(j-k)z}. \]

**Lemma 2.1.1.** Let \( f_a \) be a function of form (2). Then there exist \( M_1, M_2, M_3 > 0 \) such that, for every \( z \) with \( |Re z| \geq M_3 \), the following inequalities hold.

(i) \( M_1 e^{q|Re z|} \leq |f_a(z)| \leq M_2 e^{q|Re z|} \)

(ii) \( M_1 e^{q|Re z|} \leq |f_a'(z)| \leq M_2 e^{q|Re z|} \)

(iii) \( \frac{M_1}{M_2} |f_a'(z)| \leq |f_a(z)| \leq \frac{M_2}{M_1} |f_a'(z)| \)

where \( q = \begin{cases} k & \text{if } Re z < 0 \\ n-k & \text{if } Re z > 0. \end{cases} \)

**Proof.** Note that (iii) follows from (i) and (ii). The proof of (i) and (ii) follows from the fact that

\[ |f_a(z)| = |a_n| e^{(n-k)Re z} + o(e^{(n-k)Re z}) \text{ as } Re z \to \infty \]

\[ |f_a(z)| = |a_0| e^{-kRe z} + o(e^{-kRe z}) \text{ as } Re z \to -\infty \]

and from the observation that \( f_a' \) is a function of the same (algebraic) type as \( f_a \) (see (3)). \( \square \)
The Uniformly Expanding Property

In this section we shall prove, mainly, the very important result, Proposition 2.1.2, using McMullen’s result from [21], that any map \( f_a \in \mathcal{H} \) is uniformly expanding on its Julia set.

**Proposition 2.1.2.** For every \( f \in \mathcal{H} \) there exist \( c > 0 \) and \( \gamma > 1 \) such that

\[
|(f^n)'(z)| > c\gamma^n
\]

for every \( z \in J_f \).

**Proof.** By [21, Proposition 6.1], for all \( z \in J_f \),

\[
\lim_{n \to \infty} |(f^n)'(z)| = \infty.
\]

Since \( f \) is periodic with period \( 2\pi i \) we consider

\[A = J_f \cap \{z : \text{Im } z \in [0, 2\pi]\}\]

and we let \( A_m \) denotes the open set

\[
\{z \in A : |(f^m)'(z)| > 2\}.
\]

Then by (4) \( \{A_m\}_{m \geq 1} \) is an open covering of \( A \). Moreover, it follows from Lemma 2.1.1 that there exists \( M \) such that, if \( |\text{Re } z| > M \), then \( |f'(z)| > 2 \). Therefore

\[
\{z \in A : |\text{Re } z| > M\} \subset A_1.
\]

Since \( A \cap \{z : |\text{Re } z| \leq M\} \) is a compact subset of \( A \), it follows that there exists \( k \geq 1 \) such that the family \( \{A_1, A_2, \ldots, A_k\} \) covers \( A \). It implies that, for every \( z \in A \), there exists \( k(z) \leq k \) for which \( |(f^{k(z)})(z)| > 2 \). Therefore, for every \( n > 0 \) and every \( z \in A \) we can split the trajectory \( z, f(z), \ldots, f^n(z) \) into \( l \leq \lceil \frac{n}{k} \rceil + 1 \) pieces of the form

\[
z_i, f(z_i), \ldots, f^{k(z_i) - 1}(z_i)
\]

for \( i = 1, \ldots, l - 1 \), and, for \( i = l \),

\[
z_1, f(z_l), \ldots f^l(z_l) = f^n(z),
\]
where $z_1 = z$, $z_i = f^k(z_{i-1})(z_{i-1})$ and $j$ is some integer smaller than $k$. Then

$$|(f^n)'(z)| \geq 2^{\frac{n}{k}} \Delta^{k-1},$$

where

$$\Delta = \inf_{z \in J_f} |f'(z)| \neq 0,$$

since $J_f$ contains no critical points and because of Lemma 2.1.1 (ii). It follows that

$$|(f^n)'(z)| \geq 2^{\frac{n}{k} - 1} \Delta^{k-1} = \frac{\Delta^{k-1}}{2} (2^{\frac{1}{k}})^n.$$  

\[\square\]

More Remarks on the Family $\mathcal{H}$

We observed that the family $\mathcal{H}$ is a family in the class of transcendental entire functions, which was denoted by $\text{Ent}$. Moreover, every function $f_a$ from $\mathcal{H}$ has only finitely many singular values, in other words $f_a$ has finitely many asymptotic and critical values so $\mathcal{H}$ is a family of functions which belong to the Speiser class $\mathcal{S}$ defined in Section 1.3. Moreover, for every map in $\mathcal{H}$ the Julia set is a Cantor bouquet as it was observed in [11], [23] or [9] for maps in the Speiser class with an attracting cycle.

Note also that the assumption $0 < k < \deg P_a$ implies that any map $f_a \in \mathcal{H}$ does not have a finite asymptotic value since $P_a(z)/z^k$ converges to infinity when $z$ approaches $0$ or $\infty$. If this condition is not satisfied then one of the limits is finite and it would be a finite asymptotic value of $f_a$. Even in this case, the main result from section 4.3 may be established, using the proofs from this thesis with some minor changes. We additionally assume that maps from the family $\mathcal{H}$, which we consider, satisfy the following extra-condition:

**Assumption 2.1.3.** If $z$ is a periodic point of period $m$ then $|(f^m)'(z)| \neq 0$.

Of course we rise (sic) the question if it is possible to develop a similar theory we shall present in Chapters 3 and 4 and to prove the main result of section 4.3, without this extra-condition. This problem remains open but we believe the answer is positive.
Applying Thermodynamic Formalism on the family $\mathcal{H}$ (the reader interested in Thermodynamic formalism and its connection with dynamics is refered to [24],[29],[27] or [28]) we shall prove that the Hausdorff dimension of the subset of the Julia set of such maps, consisting of the points for which the forward orbit does not escape to infinity i.e. the set

$$J^r_{fa} = J_f \setminus I_\infty(f_a),$$

where $I_\infty(f_a) = \{ z \in \mathbb{C} : \lim_{n \to \infty} f^n(z) = \infty \}$, depends real-analytically on the parameter $a \in \mathbb{C}^{n+1}$.

In order to do that we study first quasiconformal conjugacies in the family $\mathcal{H}$ and then we define Perron-Frobenius operators associated with some special potentials. The classical theorem of Hartogs will help us to prove the main tool of this thesis (see section 4.1) which will allow us to prove the main result in section 4.3 which shows that these Perron-Frobenius operators can be embedded into a family of operators which depend holomorphically on the parameter $a$ chosen from a designed open set $G \subset \mathbb{C}^{n+1}$ and then, using perturbation theory (Kato-Rellich theorem) and the results from Chapter 3, where we prove mainly that $\text{HD}(J^r_{fa}) = h$ is the unique zero of the pressure function $t \to P(t)$ for $t > 1$ and $a \in \mathbb{C}^{n+1}$, we obtain that the function $a \mapsto \text{HD}(J^r_{fa})$ is real-analytic.

It is also important to remark that the derivative $f_b^{(s)}(z)$ of a map $f_b \in \mathcal{H}$ has the expression:

$$f_b^{(s)}(z) = \sum_{j=0}^{n} b_j (j-k)^s e^{(j-k)z}$$

for every $b = (b_0, \ldots, b_n) \in C^{n+1}$ and every positive integer $s \geq 0$. Moreover observe that the derivative with respect to the variable parameter $b \in \mathbb{C}^{n+1}$ has the expression:

$$\frac{\partial f_b^{(s)}}{\partial b}(z) = \frac{\partial f_b}{\partial b}(z) = \begin{bmatrix}
\frac{\partial f_b^{(s)}}{\partial b_0}(z) \\
\frac{\partial f_b^{(s)}}{\partial b_1}(z) \\
\vdots \\
\frac{\partial f_b^{(s)}}{\partial b_j}(z) \\
\vdots \\
\frac{\partial f_b^{(s)}}{\partial b_n}(z)
\end{bmatrix} = \begin{bmatrix}
-k e^{-kz} \\
(1-k) e^{(1-k)z} \\
\vdots \\
(j-k) e^{(j-k)z} \\
\vdots \\
(n-k) e^{(n-k)z}
\end{bmatrix}.$$
Hence

\[(6) \quad \left\| \frac{\partial f^i_b}{\partial b}(z) \right\|^2 = \left\| \frac{\partial F^i_b}{\partial b}(z) \right\|^2 = \sum_{j=0}^{n} (j - k)^2 e^{2(j-k)\Re z},\]

where \(\| \cdot \|\) means the norm on \(\mathbb{C}^{n+1}\) defined by the formula

\[\| (z_0, \ldots, z_n) \| = \sqrt{\sum_{j=0}^{n} |z_j|^2}\]

for \((z_0, \ldots, z_n) \in \mathbb{C}^{n+1}\). Similarly, it is clear that

\[(7) \quad \left\| \frac{\partial f_b}{\partial b}(z) \right\|^2 = \left\| \frac{\partial F_b}{\partial b}(z) \right\|^2 = \sum_{j=0}^{n} e^{2(j-k)\Re z}.\]

2.2. Bounded Orbits and Classical Conformal Repellers.

We fix again \(a \in \mathbb{C}^{n+1}\) and we denote \(f_a\) by \(f\), \(F_a\) by \(F\) and the Julia set of \(F\) by \(J_F\).

Our goal in this section is to prove Proposition 2.2.3. In order to prove this proposition we apply the thermodynamic formalism for compact repellers.

**Definition 2.2.1.** Let \(f\) be a holomorphic function from an open subset \(V\) of \(\mathbb{C}\) into \(\mathbb{C}\) and \(J\) a compact subset of \(V\). The triplet \((J, V, f)\) is a conformal repeller if

(i) there are \(C > 0\) and \(\alpha > 1\) such that \(|(f^n)'(z)| \geq C\alpha^n\) for every \(z \in J\) and \(n \geq 1\).

(ii) \(f^{-1}(V)\) is relatively compact in \(V\) with

\[J = \bigcap_{n \geq 1} f^{-n}(V).\]

(iii) for any open set \(U\) with \(U \cap J\) not empty, there is \(n > 0\) such that

\[J \subset f^n(U \cap J).\]

It is worth noting that there are no critical points of \(f\) in \(J\).

**Conformal Repellers**

Let \((J, V, g)\) be a (mixing) conformal expanding repeller (see for example [29] for more properties). In the proof of Proposition 2.2.3, \(J = J_1(M)\) is a compact subset of \(\mathbb{C}\), limit of a finite conformal iterated function system, \(g = F\), is a holomorphic function for which \(J\)
is invariant and for which there exist $\gamma > 1$ and $c > 0$ such that, for all $n \in \mathbb{N}$ and for all $z \in J$, $|(g^n)'(z)| \geq c\gamma^n$. For $t \in \mathbb{R}$ we consider the topological pressure defined by

$$P_z(t) = \lim_{n \to \infty} \frac{1}{n} \log P_z(n, t),$$

where

$$P_z(n, t) = \sum_{y \in g^{-n}(z)} |(g^n)'(y)|^{-t}.$$

The function $P(t) = P_z(t)$ as a function of $t$ is independent of $z$, continuous, strictly decreasing, $\lim_{t \to -\infty} P(t) = +\infty$ and the following remarkable theorem holds.

**Theorem 2.2.2 (Bowen’s Formula).** Hausdorff dimension of $J$ is the unique zero of $P(t)$.

For more details and definitions concerning the thermodynamic formalism of conformal expanding repellers (initiated by Bowen and Ruelle) we refer the reader to [29] or [24].

In order to prove Proposition 2.2.3, i.e. to show that $\text{HD}(J) > 1$, we use Bowen’s formula and we observe that, from the definition of $P_z(n, t)$, it is enough to find a constant $C > 1$ such that, for all $z \in J$,

$$P_z(1, 1) \geq C. \quad (8)$$

**Proposition 2.2.3.** Let $f \in \mathcal{H}$. Then the Hausdorff dimension of the set of points in Julia set of $f$ having bounded orbit is strictly greater than 1.

**Proof.** Let $N$ be a large number, $H = \{z \in \mathbb{C} : \Re z > N\}$. Observe that there exists $U$ such that $\overline{U} \subset \{z : s - \pi < \Im z < s + \pi\}$ for some $s \in (-\pi, \pi]$, $\Re U > 0$, $f|_U$ is univalent and $f(U) = H$. Note that, since $N$ is large, by Lemma 2.1.1 there exists $\gamma_N > 1$ such that, if $\Re z \geq N$, then

$$|F'(z)| = |f'(z)| > \gamma_N. \quad (9)$$

For every $M > N$ define

$$P(M) = \{z \in \overline{U} : N \leq \Re z \leq M\}.$$
Then, for $j \in \mathbb{Z}$, let $L_j : H \to U$ be defined by the formula

$$L_j(z) = (f|_U)^{-1}(z + 2\pi ij),$$

and let

$$Q_j(M) = L_j(P(M)).$$

The set $P(M)$ and the family of functions

$$\{L_j\}_{j \in \mathcal{K}_M}$$

with

$$\mathcal{K}_M = \{j \in \mathbb{Z} : Q_j(M) \subset \text{Int}P(M)\},$$

define a finite conformal iterated function system. By $J_1(M)$ we denote its limit set. The set $J_1(M)$ is forward $F$–invariant. From (9) and from the fact that the Julia set is the closure of the set of repelling periodic points it follows that

$$J_1(M) \subset J_F.$$

Next we need a condition for $j$ which guarantees that $Q_j(M) \subset \text{Int}P(M)$ (equivalently $j \in \mathcal{K}_M$) for all $M$ large enough. Observe that

$$\mathcal{K}_M \subset \mathcal{K}_{M+1}$$

for all $M$ large enough. To prove (12), let $j \in \mathcal{K}_M$ and let $z \in Q_j(M + 1) \setminus Q_j(M)$. Note that, if we assume that $M > M_2e^{(n-k)(N+1)}$, then we can be sure that $\text{Re } z > N + 1$ ($n$ and $k$ are defined in section 2.1). Therefore, to get (12), it is enough to prove that $\text{Re } z < M + 1$. Since

$$F(Q_j(M + 1) \setminus Q_j(M)) = P(M + 1) \setminus P(M),$$

it follows from Lemma 2.1.1 that $|F'(z)| \geq \frac{M_1^2}{M_2^2}|f(z)| \geq M$ and, then,

$$Q_j(M + 1) \setminus Q_j(M) \subset B\left(z, \frac{M_2 2\pi}{M_1 M}\right) \subset B(z, 1).$$

But we know, that, for $y \in Q_j(M)$, $\text{Re } y \leq M$. This proves (12).
The next step is to prove that there exists $j_0 \in \mathbb{N}$ such that, for all $M \in \mathbb{N}$ large enough,
\begin{equation}
(13) 
\quad j_0, j_0 + 1, \ldots, e^{\lfloor M/2 \rfloor} \in K_M.
\end{equation}

Note that we can find $j_0$ such that, for every $j \geq j_0$, $\Re Q_j(M) > N$. By Lemma 2.1.1 it is enough to take
\[
 j_0 = \left\lceil \frac{M_2 e^{(n-k)N} + 2\pi}{\pi} \right\rceil.
\]

So, to prove (13) it remains to show that $j < e^{\lfloor M/2 \rfloor}$ implies
\[
\Re Q_j(M) \leq M.
\]

Striving for a contradiction, suppose that $j < e^{\lfloor M/2 \rfloor}$ and there exists $z \in Q_j(M)$ such that $\Re z > M$. Then by Lemma 2.1.1 we have
\begin{equation}
(14) 
\quad |f(z)| > M_1 e^{(n-k)M}.
\end{equation}

Since $z \in Q_j(M)$, $f(z) \in P(M) + 2\pi ij$. Then the square of the distance from zero to the upper-right corner of $P(M) + 2\pi ij$ is greater than $|f(z)|^2$, i.e.
\[
M^2 + (s + \pi + 2\pi j)^2 > |f(z)|^2.
\]

By (14) and the assumption $j < e^{\lfloor M/2 \rfloor}$, it follows that
\[
(M_1 e^{(n-k)M})^2 < M^2 + (s + \pi + 2\pi)^2 e^M.
\]

Hence we have the required contradiction since for large $M$ the inequality is false.

Finally observe that by Lemma 2.1.1, for $j \in K_M$ and $z \in Q_j(M)$, the following is true
\[
|F'(L_j(z + 2j\pi i))| \leq \frac{M_2}{M_1} |f(L_j(z + 2\pi ij))| \leq \frac{M_2}{M_1} (2j\pi + 2\pi + M).
\]

Then
\[
P_z(1, 1) = \sum_{y \in F^{-1}(z) \cap J_1(M)} \frac{1}{|F'(y)|} = \sum_{j \in K_M} |L_j'(z + 2j\pi i)| \\
\geq \sum_{j = j_0}^{e^{\lfloor M/2 \rfloor}} \frac{1}{M_1^2 (2j\pi + 2\pi + M)}.
\]
Since, if \( M \) is large enough, the right side of this inequality can be as large as we want, (8) and the proposition are proved. □

2.3. Quasiconformal Conjugacy, Mori and Koebe’s Theorems

In this section we present some analytic and geometric properties of the family \( \mathcal{H} \). We follow the analysis from [28], which in turn follows the more elaborated descriptions from [13] and [20]. As in [13] every \( f \in \mathcal{H} \subset S \) is viewed as an element of a finite dimensional complex analytic manifold \( M_f = \mathcal{H} \subset S \). In the referred paper [13] various analytical and geometrical results are proved on \( M_f \).

For the theory of quasiconformal maps in the plane we refer the reader to the books written by Lehto and Virtanen [19], Ahlfors [2], the paper of Astala [3] and the first chapter of the book by F.Gardiner and N.Lakic [16].

A sense-preserving homeomorphism \( f \) of a domain \( G \) is called quasiconformal if its maximal dilatation \( K(G) \) is finite. If \( K(G) \leq K < \infty \) then \( f \) will be called \( K \)-quasiconformal (see [19, p.16]). Following the terminology used in the conformal case we also call a quasiconformal homeomorphism a quasiconformal mapping.

The Topology of \( \mathcal{H} \)

The domain of all functions from \( \mathcal{H} \) is the non-compact complex plane, and the most natural topology of \( \mathcal{H} \) is the topology of uniform convergence on compact subset of \( \mathbb{C} \). Observe that this topology is equivalent to the Euclidean topology on \( \mathbb{C}^{n+1} \) when we identify a parameter \( a \) with the function \( f_a \). Therefore, throughout this paper we sometimes write \( a \in \mathcal{H} \) with the meaning that \( f_a \in \mathcal{H} \). Moreover, whenever we say \( b \) is close to \( a \) we mean that \( f_b \) is close to \( f_a \) as well. We also say \( b \) is sufficiently close to \( a \) whenever we need \( b \) to be chosen from a small open neighborhood of \( a \in \mathbb{C}^{n+1} \). (compare [13]).

After this short introduction on the topological structure of \( \mathcal{H} \) we can formulate a lemma which follows from the results of Eremneko and Lyubich ([13], Proposition 5, p.1016) on structural stability of maps in the Speiser class (see also [20]).
**Lemma 2.3.1.** For \( a \in \mathcal{H} \), \( f_a \) is structurally stable i.e. if \( b \) is sufficiently close to \( a \), then there exists a conjugating quasiconformal homeomorphism \( h_b : \mathbb{C} \to \mathbb{C} \) such that

\[
f_b \circ h_b = h_b \circ f_a.
\]

Moreover the map \( b \mapsto h_b(z) \) is holomorphic for every \( z \in \mathbb{C} \) and the mapping \((b, z) \mapsto h_b(z)\) is continuous. The quasiconformal constant converges to 1 as \( b \) approaches \( a \).

This is the moment when we need our extra-condition, since, if \( f_a \) has a superattracting periodic point, then \( f_a \) is not structurally stable. This property of stability of the family \( \mathcal{H} \) stated in the previous Lemma is a crucial fact. But we need to have some control over the changes resulted from the action of the quasiconformal homeomorphism in a neighborhood of \( a \). This is stated in Proposition 2.3.7. To obtain this result we need to provide some information about quasiconformal maps and give some properties of functions from \( \mathcal{H} \).

Let \( K, \alpha > 0 \). We say that a map \( h : \mathbb{C} \to \mathbb{C} \) is \((K, \alpha)\)-Hölder continuous if

\[
|h(z_1) - h(z_2)| \leq K|z_1 - z_2|^{\alpha}
\]

for all \( z_1, z_2 \in \mathbb{C} \) such that \(|z_1 - z_2| < 1\).

But what we are really interested in is the distortion of Euclidean distances under normalized \( K \)–quasiconformal maps. Let us first recall the classical theorems of Koebe and Mori. For the proof of Koebe’s theorems the reader can see [12] and for the proof of Mori’s theorem see for example [19, p.66].

**Theorem 2.3.2 (Koebe’s One-Quarter Theorem & Koebe’s Distortion Theorem).** Let \( f : B(z_0, \rho) \to \mathbb{C} \) be a univalent map. Then

\[
B(f(z_0), \frac{1}{4}|f'(z_0)|\rho) \subset f(B(z_0, \rho))
\]

Moreover, for \( 0 < \eta < 1 \) and for \( z \in S(z_0, \eta \rho) = \{z \in \mathbb{C} : |z - z_0| = \eta \rho\} \)

(i) \[
\frac{|f'(z_0)|\eta \rho}{(1+\eta)^2} < |f(z) - f(z_0)| < \frac{|f'(z_0)|\eta \rho}{(1-\eta)^2}
\]

(ii) \[
\frac{1+\eta}{1+\eta} < \left|\frac{f'(z)}{f'(z_0)}\right| < \frac{1+\eta}{1-\eta}^2
\]

(iii) \[
|\text{arg}\left(\frac{f'(z)}{f'(z_0)}\right)| \leq 2\ln\left(\frac{1+\eta}{1-\eta}\right)
\]
Theorem 2.3.3 (Mori’s theorem). Let $f$ be a $K$-quasiconformal mapping of the unit disk onto itself normalized by $f(0) = 0$. Then for every pair of points $z, w$ with $|z| < 1$ and $|w| < 1$ we have

$$|f(w) - f(z)| \leq 16|w - z|^{\frac{1}{K}}$$

The number 16 cannot be replaced by any smaller bound if the inequality is to be hold for all $K$.

Now we formulate three lemmas about functions from the family $\mathcal{H}$. The first is very similar to Lemma 2.1.1 but we bring it into attention, once again, for the sake of completeness.

**Lemma 2.3.4.** For $a \in \mathcal{H}$ there exist positive numbers $M_1, M_2, M_3$ and $r$ such that for all $b \in B(a, r)$ and for all $z \in \mathbb{C}$ with $|\text{Re } z| > M_3$ the following inequalities hold.

(i) $M_1 e^{|\text{Re } z|q(z)} \leq |f'_b(z)| \leq M_2 e^{|\text{Re } z|q(z)}$,

(ii) $M_1 e^{|\text{Re } z|q(z)} \leq |f''_b(z)| \leq M_2 e^{|\text{Re } z|q(z)}$,

(iii) $M_1 e^{|\text{Re } z|q(z)} \leq |\frac{\partial f'_b}{\partial b}(z)| = |\frac{\partial F'_b}{\partial b}(z)| \leq M_2 e^{|\text{Re } z|q(z)}$,

where

$$q(z) = \begin{cases} k & \text{if } \text{Re } z < 0 \\ n - k & \text{if } \text{Re } z > 0. \end{cases}$$

Another important observation is that we can maintain the bounds from Lemma 2.3.4 when we apply the quasiconformal homeomorphism $h_b$ to the points of $J_{f_a}$. Note the parts (iii) and (iv) follow from the equalities (6) and (7).

**Lemma 2.3.5.** For $a \in \mathcal{H}$ there exists $M_1, M_2, M_3$ and $r$ such that for all $b \in B(a, r)$ and for all $z \in \mathbb{C}$ with $|\text{Re } z| > M_3$ the following inequalities hold.

(i) $M_1 e^{|\text{Re } z|q(z)} \leq |f'_b(h_b(z))| \leq M_2 e^{|\text{Re } z|q(z)}$,

(ii) $M_1 e^{|\text{Re } z|q(z)} \leq |f''_b(h_b(z))| \leq M_2 e^{|\text{Re } z|q(z)}$,

(iii) $M_1 e^{|\text{Re } z|q(z)} \leq |\frac{\partial f'_b}{\partial b}(h_b(z))| = |\frac{\partial F'_b}{\partial b}(h_b(z))| \leq M_2 e^{|\text{Re } z|q(z)}$,

(iv) $M_1 e^{|\text{Re } z|q(z)} \leq |\frac{\partial f_b}{\partial b}(h_b(z))| = |\frac{\partial F_b}{\partial b}(h_b(z))| \leq M_2 e^{|\text{Re } z|q(z)}$. 

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where
\[
q(z) = \begin{cases} 
  k & \text{if } \text{Re } z < 0 \\
  n-k & \text{if } \text{Re } z > 0
\end{cases}
\]

Consequently, we can also generalize Proposition 2.1.2 from section 2.2 and we obtain that for a fixed parameter \(a \in \mathcal{H}\) (i.e. \(f_a \in \mathcal{H}\)), a map \(f_b \in \mathcal{H}\) is expanding on its Julia set uniformly over a small neighborhood \(B(a, r) \subset \mathcal{H}\).

**Lemma 2.3.6.** For every \(a \in \mathcal{H}\) there exist \(c > 0, \gamma > 1, r > 0\) such that, for all \(b \in B(a, r)\),
\[
| (f_b^n)'(z) | > c\gamma^n
\]
for every \(z \in J_{f_b}\).

We state now the main result of this section.

**Proposition 2.3.7.** Fix \(a \in \mathcal{H}\). For \(b\) sufficiently close to \(a\), we can choose \(h_b : \mathbb{C} \to \mathbb{C}\), the quasiconformal conjugacy homeomorphism, such that the following three properties hold.

(i) \(\sup_{z \in J_{f_a}} \left| \frac{dh_b}{db}(z) \right| \) is bounded.

(ii) \(h_b : \mathbb{C} \to \mathbb{C}\) is \((K(Q), 1/Q)\)-Hölder continuous, where \(Q\) is quasiconformal constant of \(h_b\) and \(K : [1, \infty) \to (0, \infty)\) is increasing.

(iii) For every \(z \in \mathbb{C}\) we have \(h_b(z + 2\pi i) = h_b(z) + 2\pi i\). This shows that \(h_b\) is well defined on the cylinder \(P\).

**Proof.** First we will prove (i). Let \(f_a, f_b\) be as above. Also consider \(J_{f_b}, J_{f_a}\) and \(h_b : \mathbb{C} \to \mathbb{C}\) with \(|a - b| < \varepsilon\) for a small \(\varepsilon > 0\). We need to show that
\[
\sup_{z \in J_{f_a}, b \in B(a, \varepsilon)} \left| \frac{dh_b(z)}{db} \right| < \infty
\]
By the conjugacy relation we get
\[
h_b \circ f_a(z) = f_b \circ h_b(z) \text{ for every } z \in \mathbb{C}.
\]
Therefore, for every \(n \geq 0\), we have that
\[
h_b(f_a^n(z)) = f_b^n(h_b(z)).
\]
We consider first \( z \in J_{f_a} \) a periodic point with period \( n \geq 1 \). Define the function \( f : \mathbb{C}^n \times \mathbb{C} \to \mathbb{C} \) by the formula

\[
    f(b, z) = f_b(z)
\]

Then by the conjugacy relation we obtain, for every \( b \in B(a, \varepsilon) \),

\[
    f^n(b, h_b(z)) = h_b(z)
\]

because \( f^n_a(z) = z \). Differentiating the above relation with respect to the variable \( b \), we get

\[
    D_1f^n(b, h_b(z)) + D_2f^n(b, h_b(z)) \cdot \frac{dh_b}{db}(z) = \frac{dh_b}{db}(z).
\]

Since periodic points from the Julia set are not parabolic, this implies that

\[
    \frac{dh_b}{db}(z) = \frac{D_1f^n(b, h_b(z))}{1 - D_2f^n(b, h_b(z))}.
\]

It follows then from Lemma 2.3.6, the expanding property of maps in the family \( \mathcal{H} \) on the Julia set, that if the period \( n \) of \( z \) is large enough, then

\[
    \left| \frac{dh_b}{db}(z) \right| \leq \frac{|D_1f^n(b, h_b(z))|}{|D_2f^n(b, h_b(z))| - 1} \leq 2 \left| \frac{D_1f^n(b, h_b(z))}{D_2f^n(b, h_b(z))} \right|.
\]

Let \( w \) denotes \( h_b(z) \). Then using the equality \( f^n_b(w) = f_b(f^{n-1}_b(w)) \) (which is equivalent to \( f^n(b, w) = f(b, f^{n-1}_b(w)) \)) we can estimate \( D_1 \) in terms of \( D_2 \) as follows. First write

\[
    D_1f^n(b, w) = D_1\left(f(b, f^{n-1}(b, w))\right)
\]

\[
    = D_1f(b, f^{n-1}(b, w)) + D_2f(b, f^{n-1}(b, w)) \cdot D_1f^{n-1}(b, w).
\]

Therefore, repeating these computations for \( n, n-1, \ldots, 1, 0 \), by induction and using the chain rule we obtain

\[
    D_1f^n(b, w) = \sum_{k=0}^{n-1} D_2f^k(b, f^{n-k}(b, w)) \cdot D_1f(b, f^{n-k-1}(w))
\]

With \( \partial \)-notation, for \( w = h_b(z) \), it looks like this.

\[
    D_1f^n(b, h_b(z)) = \frac{\partial f^n}{\partial b}(b, w)
\]

\[
    = \frac{\partial f^n}{\partial b}(b, f^{n-1}_b(w)) + f'_b(f^{n-1}_b(w)) \frac{\partial f^{n-1}}{\partial b}(b, w)
\]

\[
    = \sum_{k=0}^{n-1} (f^k)'(f^{n-k}(w)) \frac{\partial f}{\partial b}(b, f^{n-k-1}(w)).
\]
Then

\[
D_1 f^n(b, h_b(z)) = \frac{\partial f^n}{\partial b}(b, w) = \sum_{k=0}^{n-1} \frac{\partial f(b, f^n_{b-k}(w))}{(f^n_{b-k})'(w)} (f^n_{b-k})'(w) = \sum_{k=0}^{n-1} \frac{\partial f(b, f^n_{b-k}(w))}{(f^n_{b-k})'(w)},
\]

Next we would like to show that

\[
\left| \frac{\partial f(b, f^n_{b-k}(w))}{(f^n_{b-k})'(w)} \right|
\]

is uniformly (with respect to \(b\)) bounded from above. It is worth reminding that \(f^n_{b-k}(w) \in J_{f_b}\) and both function \(\frac{\partial f}{\partial b}(b, \cdot)\) and \(f'_b\) are periodic with period \(2\pi i\). Therefore, it is enough to prove that there exists a constant \(C\) such that

\[
(17) \quad \left| \frac{\partial f(b, z)}{(f'_b)(z)} \right| \leq C
\]

for \(b\) sufficiently close to \(a\) and for \(z \in J_{f_b} \cap \{z \in \mathbb{C} : \text{Im } z \in [0, 2\pi]\}\).

To do this we split the set \(J_{f_b} \cap \{z \in \mathbb{C} : \text{Im } z \in [0, 2\pi]\}\) into two sets, a compact one \(\{z \in J_{f_b} : x \in [-M_3, M_3] \times [0, 2\pi]\}\) and its complement. By Lemma 2.3.6

\[
C' = \sup \left\{ \frac{|\partial f(b, x)|}{|(f'_b)(x)|} : b \in B(a, \varepsilon), x \in J_{f_b}, x \in [-M_3, M_3] \times [0, 2\pi] \right\} < \infty
\]

for \(\varepsilon\) small enough. Moreover, by Lemma 2.3.4 (i) and (iii),

\[
\left| \frac{\partial f(b, x)}{(f'_b)(z)} \right| \leq \frac{M_2}{M_1}
\]

if \(|\text{Re } x| \geq M_3\). Therefore, (17) is proved with \(C = \max\{C', M_2/M_1\}\).

Note that, it follows from Lemma 2.3.6 that we can assume that \(\varepsilon > 0\) satisfies the condition

\[
\sum_{j=0}^{l} \frac{1}{|\partial f_j'(w)|} \leq \frac{1}{c(1 - (1/\gamma))}
\]

for all \(n\) and \(b \in B(a, \varepsilon)\). Then, putting (15), (16) and (17) together, we get

\[
\sup_{z \in \text{Per}, b \in B(a, \varepsilon)} \left| \frac{dh_b}{db}(z) \right| < \infty.
\]
Hence, since $\overline{\text{Per}} = J_{f_a}$ and since $b \mapsto h_b(z)$ is analytic, the part (i) follows. Next we will prove (ii). Obviously we want to use Mori’s theorem and the result obtained before. The point (i) shows, in particular, that for small $\varepsilon$

\begin{equation}
\sup_{z \in J_{f_a}, b \in B(a, \varepsilon)} |z - h_b(z)| < 1.
\end{equation}

Let $\varepsilon > 0$ be so small that for every $b \in B(a, \varepsilon)$ the maps $f_b$ are sufficiently close to $f_a$. Fix $x \in J_{f_a}$ and consider the open disk $B(x, 1)$ of radius 1 with center at $x$. Then $G_b = h_b(B(x, 1))$ is an open simply connected set for every $b \in B(a, \varepsilon)$.

Let $R_b : D(0, 1) \to G_b$ be the conformal representation (Riemann map) of $G_b$ such that $R(0) = h_b(x)$. Then the map

$$g_b = R_b^{-1} \circ h_b : B(x, 1) \to D(0, 1)$$

is a $Q$–quasiconformal homeomorphism between two disks of radius 1. Let now $\chi_x$ be a path in $J_{F_a}$ which joins $x$ and infinity. The existence of such a path is a consequence of the fact that all Fatou components are simply connected (see [23, p.90] and section 1.4). Let $\chi_x^\omega \subset \chi_x \cap \overline{B(x, 1)}$ be an arc inside $B(x, 1)$ joining $x$ with a point on the boundary $\partial B(x, 1)$ call it $\omega$. Then $h_b(\chi_x^\omega)$ is an arc joining $h_b(x)$ and $h_b(\omega) \in \partial G_b$.

Note that there exists $z \in D(0, 1)$ with $|z| = \frac{1}{2}$ and $y \in B(x, 1) \cap J_{F_a}$ such that $R_b(z) = h_b(y) \in J_{F_b}$ (or equivalently $g_b(y) = z$). From (18), for $|a - b| < \varepsilon$, it follows that

$$|R_b(z) - R_b(0)| = |h_b(y) - h_b(x)|$$

$$\leq |h_b(y) - y| + |y - x| + |x - h_b(x)|$$

$$= |h_b(y) - h_a(y)| + |y - x| + |x - h_b(x)|$$

$$\leq 2 \varepsilon \sup \left\{ \left| \frac{\partial h_b}{\partial b} \right| : z \in J_{f_a}, b \in B(a, \varepsilon) \right\} + 1$$

$$\leq 2 \varepsilon + 1.$$ 

It follows that $R_b(B(0, 1/2))$ does not contain the ball $B(h_b(x), 2 \varepsilon + 1)$ since $R_b(B(0, \frac{1}{2}))$ does not contain any ball centered at $h_b(x)$ with radius greater than $|R_b(z) - h_b(x)|$. Then, using Koebe’s Distortion Theorem, we get $|R_b'(0)| \leq 4(2 \varepsilon + 1)$.

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Applying Mori’s Theorem to the quasiconformal mapping $g_b$ and to points $z_1, z_2 \in B(x, 1)$ we get

$$|g_b(z_1) - g_b(z_2)| < 16|z_1 - z_2|^\frac{1}{Q}.$$  

If additionally $z_1, z_2 \in B(x, 1/(32)^Q)$, then, using Koebe’s Theorem with $K = K(1/2)$ for the function $R_b$, we get

$$|h_b(z_1) - h_b(z_2)| = |R_b(g_b(z_1) - R_b(g_b(z_2))| \leq K|R'(0)||g_b(w) - g_b(z)| \leq 4K(2\varepsilon + 1)|w - z|^\frac{1}{Q}.$$  

From the above computations it follows that $h_b$ is $4K(2\varepsilon + 1), \frac{1}{Q}$-Hölder continuous on $1/(32)^Q$-neighborhood of $J_{f_a}$. But note, that there exists $r$, such that $r/(32)^Q$-neighborhood of $J_{f_a}$ contains the whole plane $\mathbb{C}$. Therefore, considering the map $g_b^*(z) = \frac{1}{z}g_b$ instead of $g_b$ (we have to increase the domain of $g_b$ to $B(x, r)$), we can repeat the computations to prove that $h_b$ is $4rK(2\varepsilon + 1), \frac{1}{Q}$-Hölder continuous on $\mathbb{C}$.

Finally we will prove (iii). Consider the map $b \mapsto k_z(b) = h_b(z + 2\pi i) - h_b(z) \in \mathbb{C}$. Since $b \mapsto h_b(z)$ is continuous, the map $k_z$ is continuous as well. If $b \in B(a, \varepsilon)$ for some small $\varepsilon$, as before, we get from the conjugacy relation that

$$(19) \quad f_b(h_b(z + 2\pi i)) = h_b(f_b(z + 2\pi i)) = h_b(f_b(z)) = f_b(h_b(z)).$$

Then, for every $b \in B(a, \varepsilon)$, the set of all possible values of $k_z$ is a discret subset of $\mathbb{C}$ (in particular has a finite intersection with the stripe $\{z : \text{Im } z \in [0, 2\pi]\}$). If $h_b(z + 2\pi i)$ and $h_b(z)$ are regular points of $f_b$, then, for $c$ sufficiently close to $b$, $h_c(z + 2\pi i)$ and $h_c(z)$ are regular points of $f_c$ and, if $k_z(b) \neq 2\pi i$, then $k_z(c) \neq 2\pi i$, and if $k_z(b) = 2\pi i$, then $k_z(c) = 2\pi i$. Since $k_z(a) = 2\pi i$ for $z \in \mathbb{C}$ and since the set of critical points of $f_a$ is discrete, $k_z$ is the constant function $2\pi i$. This finishes the proof.  

□
BIBLIOGRAPHY


